Exercise 16E: Solutions

1 a

$$\begin{aligned}A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{4\times 1 - 1\times 3}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}\end{aligned}$$

b

$$\begin{aligned}A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3\times(-4) - 2\times 1}\begin{bmatrix} -4 & -1 \\ -2 & 3 \end{bmatrix} \\ &= -\frac{1}{14}\begin{bmatrix} -4 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{7} & \frac{1}{14} \\ \frac{1}{7} & -\frac{3}{14} \end{bmatrix}.\end{aligned}$$

c

$$\begin{aligned}A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{0\times 4 - 3\times(-2)}\begin{bmatrix} 4 & -3 \\ 2 & 0 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 4 & -3 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}\end{aligned}$$

d

$$\begin{aligned}A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{(-1)\times 5 - 3\times(-4)}\begin{bmatrix} 5 & -3 \\ 4 & -1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 5 & -3 \\ 4 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}\end{aligned}$$

2 a Since the matrix of this linear transformation is

$$A = \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix},$$

the inverse transformation will have matrix

$$\begin{aligned}A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{5\times(-1) - (-2)\times 2}\begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix} \\ &= \frac{1}{-1}\begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & -5 \end{bmatrix}$$

Therefore the rule of the inverse transformation is $(x, y) \to (x - 2y, 2x - 5y)$

b Since the matrix of this linear transformation is

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},$$

the inverse transformation will have matrix

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{1 \times 0 - (-1) \times 1} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

Therefore, the rule of the inverse transformation is $(x, y) \to (y, -x + y)$.

3 a We need to solve the following equation for X.

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} X &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ X &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, $(-1, 1) \to (1, 1)$.

b We need to solve the following equation for X.

$$\begin{aligned} \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} X &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ X &= \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \end{aligned}$$

Therefore $(-\frac{1}{2}, 1) \to (1, 1)$.

4 We need to find a matrix A such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } A \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This can be written as a single equation, which we then solve to give

$$\begin{aligned} A \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \\ A &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

5 This can be solved in one step by solving the following equation for X.

$$\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} X = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} X &= \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & -1 \end{bmatrix} \end{aligned}$$

The vertices are then given by the columns of matrix X. These are $(0,0), (-1,-2), (1,1)$ and $(0,-1)$.

6 a The dilation matrix is

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}.$$

b The inverse transformation will have matrix

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{k \times 1 - 0 \times 0} \begin{bmatrix} 1 & -0 \\ -0 & k \end{bmatrix} \\ &= \frac{1}{k} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

This matrix will dilate each point from the y-axis by a factor of $\frac{1}{k}$.

7 a The shear matrix is

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

b The inverse transformation will have matrix

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{1 \times 1 - k \times 0} \begin{bmatrix} 1 & -k \\ -0 & 1 \end{bmatrix} \\ &= \frac{1}{1} \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

This matrix will shear each point from the x-direction by a factor of $-k$.

8 a The reflection matrix is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

b The inverse transformation will have matrix

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{1 \times (-1) - 0 \times 0} \begin{bmatrix} -1 & -0 \\ -0 & 1 \end{bmatrix} \\ &= -1 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= A.
\end{aligned}
$$

This is expected, since two reflections in the same axis will return the point (x, y) to its original position.

9 **a** The reflection matrix is

$$
\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.
$$

b The inverse transformation will have matrix

$$
\begin{aligned}
A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
&= \frac{1}{-\cos^2 \theta - \sin^2 \theta} \begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \\
&= \frac{1}{-1} \begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \\
&= A.
\end{aligned}
$$

This is expected, since two reflections in the same axis will return the point (x, y) to its original position.